# 2023-24 MATH2048: Honours Linear Algebra II

# Homework 7 Answer

Due: 2023-11-06 (Monday) 23:59

**For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.**

1. **Definitions.** Two linear operators $T$ and $U$ on a finite-dimensional vector space $V$ are called **simultaneously diagonalizable** if there exists an ordered basis $\beta$ for $V$ such that both $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Similarly, $A, B \in M_{n\times n}(F)$ are called simultaneously diagonalizable if there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

   (a) Prove that if $T$ and $U$ are simultaneously diagonalizable linear operators on a finite-dimensional vector space $V$, then the matrices $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable for any ordered basis $\beta$.

   (b) Prove that if $A$ and $B$ are simultaneously diagonalizable matrices, then $L_A$ and $L_B$ are simultaneously diagonalizable linear operators.

   **Solution.**

   (a) Since $T$ and $U$ are simultaneously diagonalizable, there exists an ordered basis $\alpha$ such that $[T]_\alpha$ and $[U]_\alpha$ are both diagonal matrices. For any ordered basis $\beta$, we have $[T]_\beta = [I_V]^\beta_\alpha[T]_\alpha[I_V]^\alpha_\beta$ and $[U]_\beta = [I_V]^\beta_\alpha[U]_\alpha[I_V]^\alpha_\beta$. Let $Q = [I_V]^\beta_\alpha$, then $Q^{-1}[T]_\beta Q = [T]_\alpha$ and $Q^{-1}[U]_\beta Q = [U]_\alpha$ are both diagonal. Therefore, $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable.

   (b) Since $A$ and $B$ are simultaneously diagonalizable, there exists invertible $Q \in M_{n\times n}(F)$ such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal. Suppose $Q = [q_1, q_2, \ldots, q_n]$ where $q_i$ is the $i$-th column of $Q$, and let $\beta = \{q_1, \ldots, q_n\}$, then $\beta$ is linearly independent and thus is an ordered basis for $F^n$. Let $\alpha$ be the standard ordered basis for $F^n$. Then $[L_A]_\alpha = A$, $[L_B]_\alpha = B$ and $[I_{F^n}]_\beta^\alpha = Q$.

$$[L_A]_\beta = [I_{F^n}]_\alpha^\beta [L_A]_\alpha [I_{F^n}]_\beta^\alpha = Q^{-1}AQ$$
$$[L_B]_\beta = [I_{F^n}]_\alpha^\beta [L_B]_\alpha [I_{F^n}]_\beta^\alpha = Q^{-1}BQ$$

are both diagonal. Therefore $L_A$ and $L_B$ are simultaneously diagonalizable..

2. (a) Prove that if $T$ and $U$ are simultaneously diagonalizable operators, then $T$ and $U$ commute (i.e., $TU = UT$).

   (b) Show that if $A$ and $B$ are simultaneously diagonalizable matrices, then $A$ and $B$ commute.

   **Solution.**

   (a) Since $T$ and $U$ are simultaneously diagonalizable, there exists an ordered basis $\beta$ such that $[T]_\beta$ and $[U]_\beta$ are both diagonal matrices. Since diagonal matrices commute, one has $[T]_\beta[U]_\beta = [U]_\beta[T]_\beta$.
   Therefore $[TU]_\beta = [T]_\beta[U]_\beta = [U]_\beta[T]_\beta = [UT]_\beta$, i.e. $TU = UT$.

   (b) Since $A$ and $B$ are simulaneously diagonalizable, there exists invertible matric $Q \in M_{n\times n}(F)$ such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal. Therefore, $AB = Q(Q^{-1}AQ)(Q^{-1}BQ)Q^{-1} = Q(Q^{-1}BQ)(Q^{-1}AQ)Q^{-1} = BA$.

3. Let $T$ be a linear operator on a finite-dimensional vector space $V$, and suppose that the distinct eigenvalues of $T$ are $\lambda_1, \ldots, \lambda_k$. Prove that

$$\text{span}(\{x \in V : x \text{ is an eigenvector of } T\}) = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots E_{\lambda_k}.$$

   **Solution.**

   - For any $v \in E_{\lambda_i} \cap (\sum_{j\neq i} E_{\lambda_j})$, one has $T(v) = \lambda_j v$, and $v = \sum_{j\neq i} v_j$ where $v_j \in E_{\lambda_j}$, $j \neq i$. Then $T(v) = \sum_{j\neq i} T(v_j) = \sum_{j\neq i} \lambda_j v_j$.
     Therefore, $0_V = T(v) - T(v) = \lambda_i(\sum_{j\neq i} v_j) - \sum_{j\neq i} \lambda_j v_j = \sum_{j\neq i}(\lambda_i - \lambda_j)v_j$.
     Note that $(\lambda_i - \lambda_j)v_j \in E_{\lambda_j}$, and $\lambda_i \neq \lambda_j$, $i \neq j$. One has $(\lambda_j - \lambda_i)v_j = 0_V$ and thus $v_j = 0$ for $j = 1, ..., k$. Thus $v = 0$ and $\sum_{i=1}^k E_{\lambda_i} = \oplus_{i=1}^k E_{\lambda_i}$.
   - Since $E_{\lambda_j} \subset \text{span}(\{x \in V : x \text{ is an eigenvector of } T\})$, one has $\oplus_{j=1}^k E_{\lambda_j} \subset \text{span}(\{x \in V : x \text{ is an eigenvector of } T\})$.
   - For any $v \in \text{span}(\{x \in V : x \text{ is an eigenvector of } T\})$, there exist $v_j$ such that $v = \alpha_1 v_1 + \cdots + \alpha_p v_p$ where $v_j$ is in one of these eigenspaces. After grouping

$v_1, ..., v_p$ by their eigenvalues, we have $v = w_1 + ... + w_k$ where $w_j$ is the linear combination of some $v_i$ whose eignevalue are $\lambda_j$. Therefore, $w_j \in E_{\lambda_j}$ and $v \in \oplus_{j=1}^{k} E_{\lambda_j}$.

4. Let $T$ be a linear operator on a vector space $V$, let $v$ be a nonzero vector in $V$, and let $W$ be the $T$-cyclic subspace of $V$ generated by $v$.

   (a) For any $w \in V$, prove that $w \in W$ if and only if there exists a polynomial $g(t)$ such that $w = g(T)(v)$.

   (b) Prove that the polynomial $g(t)$ in (a) can always be chosen so that its degree is less than or equal to $\dim(W)$.

   **Solution.** $W = span(\{v, T(v), ..., T^k(v), ...\})$, where $v \neq 0$.

   (a) If $w \in W$, then there exist $a_0, ..., a_k \in F$ such that $w = \sum_{i=0}^{k} a_i T^i(v)$. Let $g(t) = \sum_{i=0}^{k} a_i t^i$, then $w = g(T)(v)$.
   If $w = g(T)(v)$ for some polynomial $g$. Since $W$ is $T$-invariant, one has $W$ is $g(T)$-invariant. Then $w = g(T)(v) \in W$

   (b) Let $k = \dim(W)$, then $W = span(\{v, T(v), ..., T^{k-1}(v)\}$ and $\{v, T(v), ..., T^{k-1}(v)\}$ is a basis for $W$. Then for any $w \in W$, there exist $b_0, ..., b_{k-1} \in F$ such that $w = \sum_{j=0}^{k-1} b_j T^j(v) = g(T)(v)$ where $g(t) = \sum_{j=0}^{k-1} b_j t^j \in P_{k-1}(F)$.

5. Let A be an $n \times n$ matrix. Prove that $\dim(\text{span}(\{I_n, A, A^2, \dots\})) \leq n$.

   **Solution.** Let $f_A(t)$ be the characteristic polynomial of $A$, $f_A(t) = \det(A - tI_n) = (-1)^n t^n + a_{n-1}t^{n-1} + ... + a_0$.
   By Cayley-Hamilton theorem, one has $f_A(A) = O$. That is $(-1)^n A^n + a_{n-1}A^{n-1} + ... + a_0 I_n = O$ and $A^n \in span(\{I_n, A, ..., A^{n-1}\})$.
   Note that $A^{n+1} = A * A^n = A(-1)^{(n+1)}(a_{n-1}A^{n-1} + ... + a_0 I_n) = (-1)^{(n+1)}(a_{n-1}A^n + ... + a_0 A) \in span(\{A, A^2, ..., A^n\}) \subset span(\{I_n, A, ..., A^{n-1}\})$.
   By induction, one has $A^m \in span(\{I_n, A, ..., A^{n-1}\})$ for any $m \geq n$.
   Therefore, $W = span(\{I_n, A, ..., A^{n-1}\})$, and thus $\dim(W) \leq n$.